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APPROXIMATE ADDITIVE-QUADRATIC MAPPINGS AND BI-JENSEN MAPPINGS IN 2-BANACH SPACES

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Abstract. In this paper, we obtain the stability of the additive-quadratic functional equation

$$f(x+y,z+w)+f(x+y,z-w)=2f(x,z)+2f(x,w)+2f(y,z)+2f(y,w)$$

and the bi-Jensen functional equation

$$4f\left(\frac{x+y}{2},\frac{z+w}{2}\right)=f(x,z)+f(x,w)+f(y,z)+f(y,w)$$

in 2-Banach spaces.

1. Introduction

In 1940, Ulam [10] suggested the stability problem of functional equations concerning the stability of group homomorphisms: Let a group G and a metric group H with the metric ρ be given. For each $\varepsilon > 0$, the question is whether or not there is a $\delta > 0$ such that if $f : G \to H$ satisfies $\rho\big(f(xy), f(x)f(y)\big) < \delta$ for all $x, y \in G$, then there exists a group homomorphism $h : G \to H$ satisfying $\rho(f(x), h(x)) < \varepsilon$ for all $x \in G$.

The stability for functional equations has been investigated by a number of authors [2, 3, 6].

We introduce some definitions on 2-Banach spaces [4, 5].

Definition 1.1. Let X be a real linear space with $\dim X \geq 2$ and $\|\cdot,\cdot\| : X^2 \to \mathbb{R}$ be a function. Then $(X, \|\cdot,\cdot\|)$ is called a *linear 2-normed space* if the following conditions hold:

(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(b) $\|x, y\| = \|y, x\|$,
(c) $\|\alpha x, y\| = |\alpha|\|x, y\|$,
(d) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

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for all $\alpha \in \mathbb{R}$ and $x, y, z \in X$. In this case, the function $\|\cdot, \cdot\|$ is called a 2-*norm* on X.

Definition 1.2. Let $\{x_n\}$ be a sequence in a linear 2-normed space X. The sequence $\{x_n\}$ is said to *convergent* in X if there exits an element $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

for all $y \in X$. In this case, we say that a sequence $\{x_n\}$ converges to the limit x, simply dented by $\lim_{n\to\infty} x_n = x$.

Definition 1.3. A sequence $\{x_n\}$ in a linear 2-normed space X is called a *Cauchy sequence* if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\|x_m - x_n, y\| < \varepsilon$ for all $y \in X$. For convenience, we will write $\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$ for a Cauchy sequence $\{x_n\}$. A 2-*Banach space* is defined to be a linear 2-normed space in which every Cauchy sequence is convergent.

In the following lemma, we obtain some basic properties in a linear 2-normed space which will be used to prove the stability results.

Lemma 1.4. ([2]) *Let $(X, \|\cdot, \cdot\|)$ be a linear 2-normed space and $x \in X$.*

(a) *If $\|x, y\| = 0$ for all $y \in X$, then $x = 0$.*
(b) $\big|\|x, z\| - \|y, z\|\big| \leq \|x - y, z\|$ *for all $x, y, z \in X$.*
(c) *If a sequence $\{x_n\}$ is convergent in X, then*

$$\lim_{n\to\infty} \|x_n, y\| = \|\lim_{n\to\infty} x_n, y\|$$

for all $y \in X$.

Throughout this paper, let X be a normed space and Y be a 2-Banach space. We introduce the definitions of additive-quadratic mappings and bi-Jensen mappings.

Definition 1.5. [8] A mapping $f : X \times X \to Y$ is called a *additive-quadratic* if f satisfies the system of equations

$$\begin{aligned} f(x+y, z) &= f(x, z) + f(y, z), \\ f(x, y+z) + f(x, y-z) &= 2f(x, y) + 2f(x, z). \end{aligned} \tag{1.1}$$

Definition 1.6. [1] A mapping $f : X \times X \to Y$ is called a *bi-Jensen mapping* if f satisfies the system of equations

$$\begin{aligned} 2f\left(\tfrac{x+y}{2}, z\right) &= f(x, z) + f(y, z), \\ 2f\left(x, \tfrac{y+z}{2}\right) &= f(x, y) + f(x, z). \end{aligned} \tag{1.2}$$

For a mapping $f : X \times X \to Y$, consider the functional equations:

$$\begin{aligned} (1.3) \qquad & f(x+y, z+w) + f(x+y, z-w) \\ & = 2f(x,z) + 2f(x,w) + 2f(y,z) + 2f(y,w) \end{aligned}$$

and

$$(1.4) \qquad 4f\left(\frac{x+y}{2}, \frac{z+w}{2}\right) = f(x,z) + f(x,w) + f(y,z) + f(y,w).$$

When $X = Y = \mathbb{R}$, the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(x,y) := axy^2$ and $f(x,y) := axy + bx + cy + d$ are solutions of (1.3) and (1.4), respectively.

In 2005, W.-G. Park, J.-H. Bae and B.-H. Chung [8] obtained the general solution of (1.1) and (1.3) as follows.

Theorem 1.7. *A mapping $f : X \times X \to Y$ satisfies* (1.1) *if and only if there exist a multi-additive mapping $M : X \times X \times X \to Y$ such that $f(x,y) = M(x,y,y)$ and $M(x,y,z) = M(x,z,y)$ for all $x, y, z \in X$.*

Theorem 1.8. *A mapping $f : X \times X \to Y$ satisfies* (1.1) *if and only if it satisfies* (1.3).

In 2006, J.-H. Bae and W.-G. Park [1] obtained the general solution of (1.2) and (1.4) as follows.

Theorem 1.9. *A mapping $f : X \times X \to Y$ satisfies* (1.2) *if and only if there exist a bi-additive mapping $B : X \times X \to Y$ and two additive mappings $A, A' : X \to Y$ such that $f(x,y) = B(x,y) + A(x) + A'(y) + f(0,0)$ for all $x, y \in X$.*

Theorem 1.10. *A mapping $f : X \times X \to Y$ satisfies* (1.2) *if and only if it satisfies* (1.4).

In 2011, W.-G. Park [7] investigate approximate additive, Jensen and quadratic mappings in 2-Banach spaces. In this papaer, we also investigate additive-quadratic mappings and bi-Jensen mappings in 2-Banach spaces with different assumptions from [7].

2. Approximate additive-quadratic mappings

We obtain a result on the stability of (1.1) in 2-Banach spaces as follows.

Theorem 2.1. *Let $\varphi : X^5 \to [0,\infty)$ and $\psi : X^5 \to [0,\infty)$ be two functions satisfying*

(2.1)

$$\tilde{\varphi}(x,y,z,u,v) := \sum_{j=0}^{\infty}\left[\frac{1}{2^{j+1}}\varphi(2^j x, 2^j y, z, u, v) + \frac{1}{4^j}\varphi(x,y,2^j z,u,v)\right] < \infty$$

and

(2.2)

$$\tilde{\psi}(x,y,z,u,v) := \sum_{j=0}^{\infty}\left[\frac{1}{4^{j+1}}\psi(x, 2^j y, 2^j z, u, v) + \frac{1}{2^j}\psi(2^j x,y,z,u,v)\right] < \infty$$

for all $x,y,z,u,v \in X$. And let $f : X \times X \to Y$ be a surjective mapping such that

$$\|f(x+y,z) - f(x,z) - f(y,z), f(u,v)\| \le \varphi(x,y,z,u,v) \tag{2.3}$$

$$\|f(x,y+z) + f(x,y-z) - 2f(x,y) - 2f(x,z), f(u,v)\| \le \psi(x,y,z,u,v) \tag{2.4}$$

and $f(x,0) = 0$ for all $x,y,z,u,v \in X$. Then there exist two additive-quadratic mappings $F_a, F_q : X \times X \to Y$ such that

$$\|f(x,y) - F_a(x,y), w\| \le \tilde{\varphi}(x,x,y,u,v) \tag{2.5}$$

$$\|f(x,y) - F_q(x,y), w\| \le \tilde{\psi}(x,y,y,u,v) \tag{2.6}$$

for all $x,y,u,v \in X$, where $w = f(u,v)$.

Proof. Putting $x = y$ in (2.3), we have

$$\left\|f(x,z) - \frac{1}{2}f(2x,z), f(u,v)\right\| \le \frac{1}{2}\varphi(x,x,z,u,v) \tag{2.7}$$

for all $x,z,u,v \in X$. Thus we gain

$$\left\|\frac{1}{2^j}f(2^j x,z) - \frac{1}{2^{j+1}}f(2^{j+1}x,z), f(u,v)\right\| \le \frac{1}{2^{j+1}}\varphi(2^j x, 2^j x, z,u,v)$$

for all $x,z,u,v \in X$. Replacing z by y, we get

$$\left\|\frac{1}{2^j}f(2^j x,y) - \frac{1}{2^{j+1}}f(2^{j+1}x,y), f(u,v)\right\| \le \frac{1}{2^{j+1}}\varphi(2^j x, 2^j x, y,u,v)$$

for all $x, y, u, v \in X$. For given integer $l, m (0 \le l < m)$, we have

$$\left\| \frac{1}{2^l} f(2^l x, y) - \frac{1}{2^m} f(2^m x, y), f(u, v) \right\| \le \sum_{j=l}^{m-1} \frac{1}{2^{j+1}} \varphi(2^j x, 2^j x, y, u, v) \tag{2.8}$$

for all $x, y, u, v \in X$. By (2.1), the sequence $\{\frac{1}{2^j} f(2^j x, y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{2^j} f(2^j x, y)\}$ converges for all $x, y \in X$. Define $F_a : X \times X \to Y$ by

$$F_a(x, y) := \lim_{j \to \infty} \frac{1}{2^j} f(2^j x, y)$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (2.8), one can obtain the inequality (2.5). By (2.3) and (2.4), we obtain

$$\begin{aligned} \frac{1}{2^j} \left\| f(2^j x + 2^j y, z) - f(2^j x, z) - f(2^j y, z), f(u, v) \right\| \\ \le \frac{1}{2^j} \varphi(2^j x, 2^j y, z, u, v) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2^j} \left\| f(2^j x, y + z) + f(2^j x, y - z) - 2f(2^j x, y) - 2f(2^j x, z), f(u, v) \right\| \\ \le \frac{1}{2^j} \psi(2^j x, y, z, u, v) \end{aligned} \tag{2.9}$$

for all $x, y, z, u, v \in X$ and all integer j. Letting $j \to \infty$ and using (2.1) and (2.2), we see that F_a is additive-quadratic.

Next, setting $y = z$ in (2.4), we obtain

$$\left\| f(x, y) - \frac{1}{4} f(x, 2y), f(u, v) \right\| \le \frac{1}{4} \psi(x, y, y, u, v) \tag{2.10}$$

for all $x, y, u, v \in X$. By the same method as above, F_q is additive-quadratic which satisfies (2.6), where $F_q(x, y) := \lim_{j \to \infty} \frac{1}{4^j} f(x, 2^j y)$ for all $x, y \in X$. □

We obtain a result on the stability of (1.3) in 2-Banach spaces as follows.

THEOREM 2.2. *Let $\varphi : X^6 \to [0, \infty)$ be a function satisfying*

$$\tilde{\varphi}(x, y, z, w, u, v) := \sum_{j=0}^{\infty} \frac{1}{8^{j+1}} \varphi(2^j x, 2^j y, 2^j z, 2^j w, u, v) < \infty \tag{2.11}$$

for all $x, y, z, w, u, v \in X$. And let $f : X \times X \to Y$ be a surjective mapping such that

$$\tag{2.12} \begin{aligned} &\|f(x+y, z+w) + f(x+y, z-w) - 2f(x,z) \\ &\qquad - 2f(x,w) - 2f(y,z) - 2f(y,w), f(u,v)\| \le \varphi(x,y,z,w,u,v) \end{aligned}$$

and $f(x, 0) = 0$ for all $x, y, z, w, u, v \in X$. Then there exists a unique additive-quadratic mapping $F : X \times X \to Y$ such that

$$\tag{2.13} \|f(x,y) - F(x,y), f(u,v)\| \le \tilde{\varphi}(x,x,y,y,u,v)$$

for all $x, y, u, v \in X$.

Proof. Putting $x = y, z = w$ in (2.12), we have

$$\left\| f(x,z) - \frac{1}{8} f(2x, 2z), f(u,v) \right\| \le \frac{1}{8} \varphi(x,x,z,z,u,v)$$

for all $x, z, u, v \in X$. Thus

$$\begin{aligned} &\left\| \frac{1}{8^j} f(2^j x, 2^j z) - \frac{1}{8^{j+1}} f(2^{j+1}x, 2^{j+1}z), f(u,v) \right\| \\ &\qquad \le \frac{1}{8^{j+1}} \varphi(2^j x, 2^j x, 2^j z, 2^j z, u, v) \end{aligned}$$

for all $x, z, u, v \in X$. Replacing z by y in the above inequality, we get

$$\begin{aligned} &\left\| \frac{1}{8^j} f(2^j x, 2^j y) - \frac{1}{8^{j+1}} f(2^{j+1}x, 2^{j+1}y), f(u,v) \right\| \\ &\qquad \le \frac{1}{8^{j+1}} \varphi(2^j x, 2^j x, 2^j y, 2^j y, u, v) \end{aligned}$$

for all $x, y, u, v \in X$. For given integers $l, m (0 \le l < m)$,

$$\tag{2.14} \begin{aligned} &\left\| \frac{1}{8^l} f(2^l x, 2^l y) - \frac{1}{8^m} f(2^m x, 2^m y), f(u,v) \right\| \\ &\qquad \le \sum_{j=l}^{m-1} \frac{1}{8^{j+1}} \varphi(2^j x, 2^j x, 2^j y, 2^j y, u, v) \end{aligned}$$

for all $x, y, u, v \in X$. By (2.14), the sequence $\{\frac{1}{8^j} f(2^j x, 2^j y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{8^j} f(2^j x, 2^j y)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by

$$F(x,y) := \lim_{j \to \infty} \frac{1}{8^j} f(2^j x, 2^j y)$$

for all $x \in X$.

By (2.12), we obtain

$$\begin{aligned}\frac{1}{8^j}\Big\| & f\big(2^j(x+y), 2^j(z+w)\big) + f\big(2^j(x+y), 2^j(z-w)\big) - 2f(2^jx, 2^jz) \\ & - 2f(2^jx, 2^jw) - 2f(2^jy, 2^jz) - 2f(2^jy, 2^jw), f(u,v)\Big\| \\ \le & \frac{1}{8^j}\varphi(2^jx, 2^jy, 2^jz, 2^jw, u, v)\end{aligned}$$

for all $x, y, z, w, u, v \in X$. Letting $j \to \infty$ and using (2.11), we see that F satisfies (1.3). By Theorem 1.8, F is additive-quadratic. Setting $l = 0$ and taking $m \to \infty$ in (2.14), one can obtain the inequality (2.13). If $G : X \times X \to Y$ is another additive-quadratic mapping satisfying (2.13),

$$\begin{aligned}\|F(x,y) & - G(x,y), f(u,v)\| \\ & = \frac{1}{8^n}\|F(2^nx, 2^ny) - G(2^nx, 2^ny), f(u,v)\| \\ & \le \frac{1}{8^n}\|F(2^nx, 2^ny) - f(2^nx, 2^ny), f(u,v)\| \\ & \qquad + \frac{1}{8^n}\|f(2^nx, 2^ny) - G(2^nx, 2^ny), f(u,v)\| \\ & \le \frac{2}{8^n}\tilde{\varphi}(2^nx, 2^nx, 2^ny, 2^ny, u, v) \to 0 \text{ as } n \to \infty\end{aligned}$$

for all $x, y, u, v \in X$. Hence the mapping F is the unique additive-quadratic mapping, as desired. □

3. Approximate bi-Jensen mappings

We obtain a result on the stability of (1.2) in 2-Banach spaces as follows.

THEOREM 3.1. *Let $\varphi : X^5 \to [0,\infty)$ and $\psi : X^5 \to [0,\infty)$ be two functions such that*

$$\begin{aligned}(3.1)\quad & \tilde{\varphi}(x, y, z, u, v) \\ & := \sum_{j=0}^{\infty} \frac{1}{3^{j+1}}\left[\varphi(3^jx, 3^jy, z, u, v) + \varphi(x, y, 3^jz, u, v)\right] < \infty\end{aligned}$$

and

$$(3.2)\quad \begin{aligned}&\tilde{\psi}(x,y,z,u,v)\\ &:= \sum_{j=0}^{\infty} \frac{1}{3^{j+1}} \left[\psi(x, 3^j y, 3^j z, u, v) + \psi(3^j x, y, z, u, v)\right] < \infty\end{aligned}$$

for all $x, y, z, u, v \in X$. And let $f : X \times X \to Y$ be a mapping such that

$$(3.3)\quad \left\|2f\left(\frac{x+y}{2}, z\right) - f(x,z) - f(y,z), f(u,v)\right\| \le \varphi(x,y,z,u,v)$$

$$(3.4)\quad \left\|2f\left(x, \frac{y+z}{2}\right) - f(x,y) - f(x,z), f(u,v)\right\| \le \psi(x,y,z,u,v)$$

for all $x, y, z, u, v \in X$. Then there exist two bi-Jensen mappings F, $F' : X \times X \to Y$ such that

(3.5)
$$\|f(x,y) - f(0,y) - F(x,y), w\| \le \tilde{\varphi}(x,-x,y,u,v) + \tilde{\varphi}(-x,3x,y,u,v),$$
(3.6)
$$\|f(x,y) - f(x,0) - F'(x,y), w\| \le \tilde{\psi}(x,y,-y,u,v) + \tilde{\psi}(x,-y,3y,u,v)$$

for all $x, y, u, v \in X$, where $w = f(u,v)$.

Proof. Letting $y = -x$ in (3.3) and replacing x by $-x$ and y by $3x$ in (3.3), one can obtain that

$$\|2f(0,z) - f(x,z) - f(-x,z), f(u,v)\| \le \varphi(x,-x,z,u,v),$$

$$\|2f(x,z) - f(-x,z) - f(3x,z), f(u,v)\| \le \varphi(-x,3x,z,u,v),$$

respectively, for all $x, z, u, v \in X$. By the above two inequalities and replacing z by y, we get

$$\begin{aligned}&\|3f(x,y) - 2f(0,y) - f(3x,y), f(u,v)\|\\ &\quad \le \varphi(x,-x,y,u,v) + \varphi(-x,3x,y,u,v)\end{aligned}$$

for all $x, y, u, v \in X$. Thus we have

$$\begin{aligned}&\left\|\frac{1}{3^j} f(3^j x, y) - \frac{2}{3^{j+1}} f(0,y) - \frac{1}{3^{j+1}} f(3^{j+1}x, y), f(u,v)\right\|\\ &\le \frac{1}{3^{j+1}} \left[\varphi(3^j x, -3^j x, y, u, v) + \varphi(-3^j x, 3^{j+1}x, y, u, v)\right]\end{aligned}$$

for all $x, y, u, v \in X$ and all j. For given integer $l, m(0 \leq l < m)$, we obtain

$$\left\| \frac{1}{3^l} f(3^l x, y) - \sum_{j=l}^{m-1} \frac{2}{3^{j+1}} f(0, y) - \frac{1}{3^m} f(3^m x, y), f(u, v) \right\| \tag{3.7}$$
$$\leq \sum_{j=l}^{m-1} \frac{1}{3^{j+1}} \left[\varphi(3^j x, -3^j x, y, u, v) + \varphi(-3^j x, 3^{j+1} x, y, u, v) \right]$$

for all $x, y, u, v \in X$. By (3.1), the sequence $\{\frac{1}{3^j} f(3^j x, y)\}$ is a Cauchy sequence for all $x, y \in X$. Since Y is complete, the sequence $\{\frac{1}{3^j} f(3^j x, y)\}$ converges for all $x, y \in X$. Define $F : X \times X \to Y$ by

$$F(x, y) := \lim_{j \to \infty} \frac{1}{3^j} f(3^j x, y)$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (3.7), one can obtain the inequality (3.5). By (3.3), we get

$$\left\| \frac{2}{3^j} f\left(\frac{3^j(x+y)}{2}, y \right) - \frac{1}{3^j} f(3^j x, y) - \frac{1}{3^j} f(3^j y, z), f(u, v) \right\|$$
$$\leq \frac{1}{3^j} \varphi(3^j x, 3^j y, y, u, v)$$

for all $x, y, z, u, v \in X$ and all j. By (3.4), we have

$$\left\| \frac{2}{3^j} f\left(3^j x, \frac{y+z}{2} \right) + \frac{1}{3^j} f(3^j x, y) - \frac{1}{3^j} f(3^j x, z), f(u, v) \right\|$$
$$\leq \frac{1}{3^j} \psi(3^j x, y, z, u, v)$$

for all $x, y, z, u, v \in X$ and all j. Letting $j \to \infty$ in the above two inequalities and using (3.1) and (3.2), F is a bi-Jensen mapping.

Define $F' : X \times X \to Y$ by

$$F'(x, y) := \lim_{j \to \infty} \frac{1}{3^j} f(x, 3^j y)$$

for all $x, y \in X$. By the same method in the above argument, F' is a bi-Jensen mapping satisfying (3.6). $\square$

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